J. Fluid Mech. (1973), *vol.* 59, *part* 2, *pp.* 257–262
Printed in Great Britain

The singularity at the crest of a finite amplitude progressive Stokes wave

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(Received 15 May 1972 and in revised form 22 March 1973)

Expansions have been given in the past for steady Stokes waves at or near a largest wave with a 120° corner. It is shown here that the solution is more complicated than has been assumed: that the corner is not a regular singular point, and that waves of less than maximum amplitude have singularities of a different order.

1. Introduction

The irrotational motion under gravity of a deep heavy inviscid fluid has been studied since the last century. Despite the problem's simple formulation, it has not been completely solved. The difficulty is the awkward nonlinearity of the conditions at the water's surface.

Most work on permanent surface waves has been concerned with steady progressing two-dimensional waves. This is surveyed in Wehausen & Laitone (1960). Small amplitude waves are sinusoidal. It is possible to construct an expansion in powers of amplitude, the Stokes expansion. The higher order terms become more important as the amplitude grows, and the wave profile changes. The crests become steeper, the troughs shallower. Stokes showed that the only possible singularity of a steady flow is a corner of 120°. It is a natural conclusion to assume that waves of increasing amplitude change shape until a largest wave is reached which has a 120° corner and a wide shallow trough. There is no proof that such a wave exists, although Krasovskii (1962) proved that waves exist with maximum slopes up to, but not including, 30°. A wave with a 120° corner would have slopes of $\pm 30°$ at the corner. This is fairly strong evidence that the wave with a 120° corner also exists, and it has generally been assumed that it does, even though large amplitude gravity waves show capillary-scale ripples near the crest, rather than a corner.

Several people have tried to compute the shape of this wave, such as Michell (1893), Nekrasov (1920) and Yamada (1957). Longuet-Higgins (1973) gave a one-term approximation that fits closely the previously calculated profile. When the problem is formulated in streamline co-ordinates, a 120° corner corresponds to an analytic function $z(f)$ having a singularity of order $\frac{2}{3}$. Existing expansions all tacitly assume that this is a regular singular point. However, an attempt to expand about the corner quickly reveals that this is not regular, and the ex-

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pansion of z involves irrational powers of f. Consequently, these expansions do not adequately describe the nature of the corner, and may similarly fail to describe the wave profile.

Havelock (1919) constructed an expansion for waves of arbitrary amplitude, which assumes that the solution has the same order $\frac{2}{3}$ singularity for all amplitudes, approaching the surface at maximum amplitude. This is an attractive assumption, but incorrect, and the misrepresentation is worse than before. Instead of one order $\frac{2}{3}$ singularity above the fluid, there are several order $\frac{1}{2}$ singularities which coalesce at maximum amplitude.

In general, the structure near the corner is considerably more complicated than has been assumed in the past. These results are confirmed by the numerical work of Schwartz (1972), particularly the change in order of the singularity as it leaves the surface.

2. The equations of motion

The equations of steady flow of a heavy inviscid fluid are well known. In a frame of reference moving with the wave profile, with units of length and time such that the wave speed and gravitational acceleration are unity, they are the Bernoulli condition

$$\tfrac{1}{2}(\nabla\phi)^2 + y = 0 \quad \text{at} \quad y = \eta, \tag{2.1}$$

the kinematic condition

$$\phi_x \eta_x - \phi_y = 0 \quad \text{at} \quad y = \eta, \tag{2.2}$$

$$\nabla^2\phi = 0, \quad y \leqslant \eta, \tag{2.3}$$

$$\phi \to x, \quad y \to -\infty, \tag{2.4}$$

where the surface is given by $y = \eta(x)$. The constant in the Bernoulli condition has been set zero by the choice of the origin. The condition at great depth implies that the fluid is moving at unit velocity with respect to the co-ordinate frame. The solution with no waves is $\phi = x$, $\eta = -\frac{1}{2}$.

Since the wave height $\eta(x)$ is one of the unknowns, the equations are transcendentally nonlinear. This can be improved by a transformation of the fluid volume to a fixed, known domain. The nonlinearity becomes 'only' polynomial. While the equations are still insoluble, manipulations are easier. For numerical work on a computer, or by hand, the decrease in storage space and effort is considerable, as the expansions of only a fixed number of powers of the dependent variables are needed. Equations (2.1) and (2.2) expanded about $\eta = -\frac{1}{2}$, involve η^n at nth order. This more convenient representation is given by transforming into streamline co-ordinates (ϕ, ψ). Then $z = x + iy$ is an analytic function of the complex potential $f = \phi + i\psi$:

$$z = z(f) \quad \text{inside the fluid.} \tag{2.5}$$

The kinematic condition (2.2) states that the surface is a streamline, which can be taken as $\psi = 0$. The fluid then occupies the lower half-space. The condition (2.4) at great depth is now

$$z \to f \quad \text{as} \quad \psi \to -\infty. \tag{2.6}$$

Reformulating the Bernoulli condition, we have

$$\tfrac{1}{2}(\nabla\phi)^2 = \frac{1}{2}\left|\frac{df}{dz}\right|^2 = \frac{1}{2}\left|\frac{dz}{df}\right|^{-2} = -\eta = -\operatorname{Im} z,$$

$$-2\operatorname{Im} z\left|\frac{dz}{df}\right|^2 = 1 \quad \text{at} \quad \psi = 0. \tag{2.7}$$

This is the same as equation (34.1) of Wehausen & Laitone (1960). The inverse transformation should also exist and be regular:

$$\frac{dz}{df} \neq 0, \quad \psi \leqslant 0. \tag{2.8}$$

If z is an analytic function of f throughout the fluid and on its surface, the wave profile must be smooth. In a limiting case, there may be a singularity on the surface, when the wave profile has a corner. The corner angle depends on the singularity: if the singularity is of order μ, the surface has a corner of angle $\pi\mu$.

For if $z(f) = z_0 + \beta(f-\phi_0)^{\mu} + o[(f-\phi_0)^{\mu}]$ near $f = \phi_0$, $0 < \mu < 2$, then

$$\arg(z-z_0) = \arg(\beta) + \mu \arg(f-\phi_0) + o(1).$$

As f moves along the surface from left to right, $f-\phi_0$ changes from real negative to real positive, and its argument increases by π. Then arg $(z-z_0)$ increases by $\pi\mu$, and the surface has a corner of angle $\pi\mu$ at $z = z_0$.

It is usual to consider only symmetric waves. The peak can be chosen to be at $\phi = 0$. Then either both, or neither, of (x, y) and $(-x, y)$ lie on the surface. That is, z lies on the surface if and only if $-z^*$ does. This implies that

$$z(-f) = -z(f^*). \tag{2.9}$$

3. Waves of greatest height

Stokes' singular solution is

$$z = -i\, e^{\frac{1}{3}\pi i}\,(\tfrac{3}{2}f)^{\frac{2}{3}}. \tag{3.1}$$

This satisfies the surface condition exactly, and is analytic inside the fluid, provided the cut is taken above the ϕ axis. The surface is given by $f = \phi$ (ϕ is real), with

$$\arg(z) = \begin{cases} -\frac{1}{6}\pi & \text{if} \quad \phi > 0, \\ -\frac{5}{6}\pi & \text{if} \quad \phi < 0. \end{cases}$$

The surface consists of two lines radiating from the point $z = 0$, with slopes of $+30°$ and $-30°$. The fluid occupies this infinite $120°$ sector, and is not a bounded travelling wave. This can be seen from the fact that it does not satisfy the condition at great depth, $z \sim f$. This solution can be only locally valid, the first term in an expansion about the corner. Higher terms in this expansion are needed to describe behaviour away from the corner. We look for the next term in the expansion

$$z = -i(\tfrac{3}{2}if)^{\frac{2}{3}} + i\alpha(if)^{\mu} + \dots \quad (\mu > \tfrac{2}{3}). \tag{3.2}$$

Inserting (3.2) into the Bernoulli condition (2.7) should determine μ. Since (3.1) is an exact solution, the term $i\alpha(if)^{\mu}$ must satisfy a linear homogeneous equation which has μ as an eigenvalue. The constant α cannot be found from local argument alone. Let $\alpha = \rho\, e^{\pi i\gamma}$, then

$$z'(f) = \tfrac{3}{2}(if)^{-\frac{1}{3}} - \alpha\mu(if)^{\mu-1} = (\tfrac{3}{2}\phi)^{-\frac{1}{3}}\exp\left[-\tfrac{1}{6}\pi i\right] - \rho\mu\phi^{\mu-1}\exp\left[\tfrac{1}{2}\pi i(\mu-1)+\pi i\gamma\right],$$

$$-2\,\mathrm{Im}\, z = (\tfrac{3}{2}f)^{\frac{2}{3}} - 2\rho\phi^{\mu}\cos\left(\tfrac{1}{2}\pi\mu+\pi\gamma\right).$$

Substituting into the surface condition and collecting terms of order $\phi^{\mu-\frac{2}{3}}$ gives

$$0 = \cos\left(\tfrac{1}{2}\pi\mu+\pi\gamma\right) + \tfrac{3}{2}\mu\cos\left(\tfrac{1}{2}\pi\mu+\pi\gamma-\tfrac{1}{3}\pi\right),$$

or

$$\tan(\pi\delta+\pi\gamma) = -(2+3\delta)/3^{\frac{3}{2}}\,\delta, \tag{3.3}$$

where $\delta = \frac{1}{2}\mu > \frac{1}{3}$. If the expansion (3.2) is to be well behaved, μ, and hence δ, should be rational. Best would be $\mu = \frac{2}{3}+\text{integer}$. Then if δ is rational,

$$\tan(\pi\delta+\delta\gamma)$$

is a negative rational number times $3^{\frac{1}{2}}$. The only such choices, for $\delta+\gamma$ rational, are $-3^{\frac{1}{2}}$ and $-1/3^{\frac{1}{2}}$. Then

$$2+3\delta = 3^{\frac{3}{2}}\,\delta(3^{\frac{1}{2}} \text{ or } 1/3^{\frac{1}{2}}) = 9\delta \quad \text{or} \quad 3\delta.$$

The second choice is contradictory, the first gives $\delta = \frac{1}{3}$. Hence it is not possible for both γ and δ to be rational.

A particular case is the usually assumed symmetric wave. Then (2.9) implies that α is real, and so μ is irrational, and probably transcendental. The expansion (3.2) is not an expansion in integral or fractional powers of f, and the corner is not a regular singular point of the function $z(f)$.

Any approximation of $z(f)$ should have a structure similar to z. Consider expansions of the form

$$z'(f) = F(f)\,S(f), \tag{3.4}$$

where $F(f)$ is a known function with a regular singular point of order $-\frac{1}{3}$ and S is an unknown function which is approximated in some way. Since z is irregular, so is S. This factoring does not regularize the problem, and there remains some singular behaviour in S. This form is a natural one, and many methods use it. For example, Michell's method has (Wehausen & Laitone 1960, equation 33.18)

$$z'(f) = c^{-1}2^{-\frac{1}{3}}\,e^{\pi if/3c\lambda}\left[i\sin\left(\frac{\pi f}{c\lambda}\right)\right]^{-\frac{1}{3}}\left[\sum_{n=0}^{\infty} b_n\, e^{-2\pi inf/c\lambda}\right]^{-1} \tag{3.5}$$

and Nekrasov and Yamada have

$$z'(f) = c^{-1}(1-e^{-2\pi if/c\lambda})^{-\frac{1}{3}}\sum_{n=0}^{\infty} a_n\, e^{-2\pi inf/c\lambda}. \tag{3.6}$$

These use dimensional co-ordinates, and dimensionless forms are found by setting $c = 1$. Neither of these expansions adequately represents the solution. For example, depending on the value of μ, some derivative of the sum S will diverge.

Equation (3.3), with $\gamma = 0$, has infinitely many roots. The first, greater than $\frac{2}{3}$, is $\mu = 1{\cdot}468$. If it is assumed that this is the relevant root, the first derivative of S will diverge. This also tells us how rapidly the surface deviates from the 30° slope:

$$\left.\begin{aligned} z &= -i\, e^{\frac{1}{3}\pi i}\,(\tfrac{3}{2}\phi)^{\frac{2}{3}} + i\alpha(i\phi)^{\mu} \\ &= e^{-\frac{1}{6}\pi i}\,(\tfrac{3}{2}\phi)^{\frac{2}{3}} + \alpha\phi^{\mu}\, e^{\frac{1}{2}\pi i(\mu+1)} \\ &= e^{-\frac{1}{6}\pi i}\, r + \alpha\, e^{\frac{1}{2}\pi i(\mu+1)}\, r^{\frac{3}{2}\mu} \end{aligned}\right\} \quad (\phi > 0).$$

The slope deviates from the value $-30°$ at the corner like $r^{\frac{3}{2}\mu-1} = r^{1\cdot 20}$, or slightly more slowly than a parabola.

4. Waves of less than maximum height

Similar problems arise with waves of less than maximum amplitude. The natural assumption to make is that the $f^{\frac{2}{3}}$ singularity has moved off the surface into the upper half-plane. This, however, is not the case. For less than maximum amplitude z has singularities only of order $\frac{1}{2}$, and the approach to maximum height is correspondingly more complicated.

Consider $z^*(f)$, the function conjugate to $z(f)$, defined by $z^*(f) = [z(f^*)]^*$. Since z is analytic and z' non-zero in the lower half-plane z^* is analytic and $z^{*\prime}$ non-zero in the upper half-plane. The surface condition (2.7) is

$$[A(f) + iz]\, z' B(f) = 1, \tag{4.1}$$

where $A(f) = -iz^*$ and $B(f) = z^{*\prime}$. This is analytic in the two analytic functions z and z^*, and so can be extended everywhere they are both analytic. Consider (4.1) in the upper half-plane. $A(f)$ is analytic, and $B(f)$ analytic and non-zero. As an equation for $z(f)$, (4.1) admits only singularities of order $\frac{1}{2}$. A different result is possible at maximum amplitude, for then the singularities of z and z^* coincide.

Now also consider the effect of increasing the amplitude. At maximum amplitude, $z(f)$ has one singularity, of order $\frac{2}{3}$. For any lesser amplitude, it has singularities only of order $\frac{1}{2}$. The only way a continuous approach to greatest amplitude is possible is for z to have several coalescing singularities.

Havelock's expansion, for waves of arbitrary amplitude, uses Michell's expansion (3.5), but with the singularity placed at $f = i\alpha$, $\alpha > 0$:

$$z'(f) = c^{-1} 2^{-\frac{1}{3}}\, e^{i\pi(f-i\alpha)/3c\lambda} \left[i \sin\left(\frac{\pi(f-i\alpha)}{c\lambda}\right)\right]^{-\frac{1}{3}} \left[\sum_{n=0}^{\infty} d_n\, e^{-2n\pi i f/c\lambda}\right]^{-1}. \tag{4.2}$$

He is thus assuming that z has one order $\frac{2}{3}$ singularity above the fluid, which moves down to touch the surface at maximum amplitude. Not only does this have singularities of the wrong order, but also the wrong number.

Schwartz (1972) numerically computed the perturbation expansion to high order, and manipulated the series to show the structure near the singularity. He found that Havelock's expansion was unsuitable because the singularity changed order, from $\frac{2}{3}$ to $\frac{1}{2}$, as it left the fluid, and that the structure in general at and near greatest amplitude was more complicated than any of these past expansions had assumed.

I am considerably indebted to Professor D. J. Benney for his help and guidance.

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